An Error Analysis of Elliptical Orbit Transfer Between Two Coplanar Circular Orbits

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Abstract

We consider transfer orbits between two coplanar, confocal circular orbits. We calculate the semimajor axis and eccentricity of the elliptical transfer orbit, as well as the energy and velocity magnitude changes required to accomplish the transfer. We assume the energy changes (i.e., thruster firings) occur over a short time interval compared to the relevant orbital periods. We then consider the consequences of small errors in the thruster firings.

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1 Motivation

Consider two coplanar, circular, confocal orbits of semimajor axes a_1 and $a_2 > a_1$. Suppose we wish to transfer an artificial satellite from the inner to the outer orbit and do it in such a way that the *transfer orbit* is elliptical (as opposed to hyperbolic or parabolic) and confocal to the circular orbits. This type of orbit was first considered by W. Hohmann in 1925. At some point during the inner circular orbit, a thruster firing perpendicular to the radius vector occurs, at which point the satellite's motion switches to the pericenter of the elliptical transfer orbit. The ellipse is such that its apocenter distance coincides with the outer circular orbit distance. After coasting from the pericenter out to the apocenter, another thruster firing (again in the orbital plane and perpendicular to the instantaneous radius vector) places the satellite on the outer circular orbit. This process is symmetric, so one can just as easily transfer from an outer orbit to an inner one, requiring only a few changes in sign in what follows. Hence, what are the semimajor axis a and eccentricity e of the transfer orbit, and what are the changes in energy and velocity magnitude required to perform the transfer?

2 Orbital Elements of the Transfer Orbit

Now, the pericenter of the transfer ellipse (call it P) coincides with the radius of the smaller circle, $a(1-e) = a_1$, and the apocenter (call it A) coincides with the radius of the larger circle, $a(1+e) = a_2$. Thus, we may combine these two relations to find the semimajor axis and eccentricity of the transfer orbit:

$$a = \frac{1}{2}(a_1 + a_2) \tag{1}$$

$$e = \frac{a_2 - a_1}{a_1 + a_2} \tag{2}$$

Thus, we have a tidy result. The semimajor axis of the transfer ellipse is the average of the radii of the circular orbits, and the eccentricity is the fractional difference of those radii.

The amount of time spent in the transfer orbit is, by definition, half the orbital period of the transfer orbit. Kepler's third law can be written

$$\mu = n^2 a^3 \tag{3}$$

where $\mu = G(m_1 + m_2)$, $n = 2\pi/T$ is the mean motion, T is the orbital period. Using (1), we therefore find

$$\frac{T}{2} = \pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{(a_1 + a_2)^3}{8\mu}} \tag{4}$$

3 Energy Considerations

The *specific energy* $\mathcal{E}$ of a two-body orbit is

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{5}$$

where v is the magnitude of the relative velocity vector, and r is the magnitude of the relative position vector. The specific energy is the energy of the two-body system, $\mathcal{E}_{\mathrm{tot}}$, divided by the reduced mass,

$$\frac{m_1 m_2}{m_1 + m_2}\mathcal{E} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v^2 - \frac{G m_1 m_2}{r} = \mathcal{E}_{\mathrm{tot}} \tag{6}$$

One can also show that

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{7}$$

which is a statement of conservation of energy known as the *vis viva integral.* Combining (5) and (7) lets us write the specific energy as

$$\mathcal{E} = \frac{-\mu}{2a} \tag{8}$$

Thus, the difference in energy (henceforth we shall assume "energy" means "specific energy") between the two circular orbits is

$$\Delta\mathcal{E} = \frac{\mu}{2}\left(\frac{1}{a_1} - \frac{1}{a_2}\right) \tag{9}$$

4 Energy Changes to Accomplish the Transfer

From (8) and (1), we can write the energy of the transfer orbit,

$$\mathcal{E} = \frac{-\mu}{a_1 + a_2} \tag{10}$$

Hence, we discover that the changes in energy that must occur at P and at A are, from (10) and (8),

$$\Delta\mathcal{E}_P = \mathcal{E} - \mathcal{E}_P = \frac{\mu}{2a_1}\frac{a_2 - a_1}{a_1 + a_2} = -e\mathcal{E}_P \tag{11}$$

and

$$\Delta\mathcal{E}_A = \mathcal{E}_A - \mathcal{E} = \frac{\mu}{2a_2}\frac{a_2 - a_1}{a_1 + a_2} = -e\mathcal{E}_A \tag{12}$$

where $\mathcal{E}_P = \frac{-\mu}{2a_1}$ and $\mathcal{E}_A = \frac{-\mu}{2a_2}$ are the energies of the respective circular orbits. It is quite delightful that the changes in energy are just the respective orbital energies scaled only by the transfer orbit eccentricity. One can easily show that (11) and (12) add up to (9):

$$\Delta\mathcal{E}_P + \Delta\mathcal{E}_A = -e(\mathcal{E}_P + \mathcal{E}_A) = \Delta\mathcal{E} \tag{13}$$

5 Changes in Relative Velocity Magnitude to Accomplish the Transfer

Let us assume that the changes in orbital energy are accomplished by thruster firings which are very short compared to the orbital periods involved. Then, since the changes occur at pericenter and apocenter of the elliptical transfer orbit, our assumptions let us put $\dot{r} = 0$, and the energy changes involve only the kinetic energies and hence the relative velocity magnitudes.

5.1 Using Physics

Differentiating (5), we have

$$\frac{d\mathcal{E}}{dt} = v\frac{dv}{dt} + \frac{\mu}{r^2}\frac{dr}{dt} = v\frac{dv}{dt} \tag{14}$$

since during the thruster firings $\dot{r} = 0$. Integrating through the thruster firing from (say) times t_0 to $t_0 + \Delta t$,

$$\int_{\mathcal{E}_0}^{\mathcal{E}_0+\Delta\mathcal{E}} d\mathcal{E} = \int_{v_0}^{v_0+\Delta v} v\,dv \tag{15}$$

we have

$$\Delta\mathcal{E} = \frac{1}{2}(v_0 + \Delta v)^2 - \frac{1}{2}v_0^2 = \frac{1}{2}\Delta v^2 + v_0\Delta v \tag{16}$$

where $v_0 = v(t_0)$ is the magnitude of the velocity the instant before the thruster firing, and Δv is the velocity magnitude impulse arising from a change in energy $\Delta\mathcal{E}$. Thus, the changes in relative velocity magnitude are

$$\Delta v = -v_0 \pm \sqrt{2\Delta\mathcal{E} + v_0^2} \tag{17}$$

at the respective orbital locations. At P,

$$v_0(P) = \sqrt{\frac{\mu}{a_1}} \tag{18}$$

which we obtained from eq. (7) for the interior circular orbit, while at A we similarly find

$$v_0(A) = \sqrt{\frac{\mu}{a}\frac{1-e}{1+e}} = \sqrt{\frac{2\mu}{a_1+a_2}\frac{a_1}{a_2}} \tag{19}$$

for the elliptical orbit. Plugging (18) and (19), and (11) and (12), into (17), we find that

$$\Delta v_P = -\sqrt{\frac{\mu}{a_1}}\left(1 \pm \sqrt{1+e}\right) = -\sqrt{\frac{\mu}{a_1}}\left(1 \pm \sqrt{\frac{2a_2}{a_1+a_2}}\right) \tag{20}$$

and

$$\Delta v_A = -\sqrt{\frac{\mu}{a_2}}\left(\sqrt{\frac{2a_1}{a_1+a_2}} \pm 1\right) \tag{21}$$

We must choose the signs in (20) and (21) to match physical circumstances. The first thruster firing is in the same direction as the direction of motion around the inner circular orbit, so the change in velocity magnitude is positive. Likewise, the second thruster firing is also along the direction of motion, so that change, too, is positive. Therefore, since we are considering the case $a_2 > a_1$, the changes in velocity magnitude are

$$\Delta v_P = \sqrt{\frac{\mu}{a_1}}\left(\sqrt{1+e} - 1\right) = \sqrt{\frac{\mu}{a_1}}\left(\sqrt{\frac{2a_2}{a_1+a_2}} - 1\right) \tag{22}$$

and

$$\Delta v_A = \sqrt{\frac{\mu}{a_2}}\left(1 - \sqrt{\frac{2a_1}{a_1+a_2}}\right) \tag{23}$$

In each case the change in velocity magnitude is a fraction of the respective circular velocity $v_c = \sqrt{\mu/a}$, the fractions being the expressions in parentheses in eqs. (22) and (23).

5.2 Using Algebra

The previous derivation uses conservation of energy as the starting point. It is therefore satisfying in that is makes use of a fundamental physical principle. If we are willing to forgo physics, then here is a short algebraic derivation. From (7) the circular velocities at P and at A are

$$v_c(P) = \sqrt{\frac{\mu}{a_1}} \qquad \text{and} \qquad v_c(A) = \sqrt{\frac{\mu}{a_2}} \tag{24}$$

The velocities on the transfer orbit at those same points are (again using (7))

$$v_P = \sqrt{\frac{\mu}{a}\frac{1+e}{1-e}} = \sqrt{\frac{2\mu}{a_1+a_2}\frac{a_2}{a_1}} \tag{25}$$

$$v_A = \sqrt{\frac{\mu}{a}\frac{1-e}{1+e}} = \sqrt{\frac{2\mu}{a_1+a_2}\frac{a_1}{a_2}} \tag{26}$$

Thus, the changes in velocity magnitude are just

$$\Delta v_P = v_P - v_c(P) \tag{27}$$

and

$$\Delta v_A = v_c(A) - v_A \tag{28}$$

which yield eqs. (22) and (23).

6 Expansions

Suppose $a_1 \ll a_2$. Then we can expand on $\varepsilon = a_1/a_2$, and (22) and (23) become

$$\Delta v_P = \sqrt{\frac{\mu}{a_1}}\left(\sqrt{2}-1\right) \; - \; \sqrt{\frac{2\mu}{a_2}}\left(\frac{1}{2}\sqrt{\varepsilon} - \frac{3}{8}\varepsilon^{3/2} + \frac{5}{16}\varepsilon^{5/2} - \cdots\right) \tag{29}$$

which we can also write as

$$\Delta v_P = \sqrt{\frac{\mu}{a_1}}\left(\sqrt{2}-1\right) \; - \; \sqrt{\frac{2\mu}{a_1}}\left(\frac{1}{2}\varepsilon - \frac{3}{8}\varepsilon^{2} + \frac{5}{16}\varepsilon^{3} - \cdots\right) \tag{30}$$

and

$$\Delta v_A = \sqrt{\frac{\mu}{a_2}} \; - \; \sqrt{\frac{2\mu}{a_2}}\left(\sqrt{\varepsilon} + \frac{1}{2}\varepsilon^{3/2} - \frac{3}{8}\varepsilon^{5/2} + \cdots\right) \tag{31}$$

which, similarly, we can write as

$$\Delta v_A = \sqrt{\frac{\mu}{a_2}} \; - \; \sqrt{\frac{2\mu}{a_1}}\left(\varepsilon + \frac{1}{2}\varepsilon^{2} - \frac{3}{8}\varepsilon^{3} + \cdots\right) \tag{32}$$

From (29) and (31) we see that the thruster firings consist of a large kick (of order the circular velocity at the corresponding radius) followed by higher order terms.

The ratio of velocity kicks is

$$\frac{\Delta v_A}{\Delta v_P} = \sqrt{\varepsilon}\frac{\sqrt{1+\varepsilon}-\sqrt{2\varepsilon}}{\sqrt{2}-\sqrt{1+\varepsilon}} \tag{33}$$

A series expansion of (33) yields

$$\begin{aligned}\frac{\Delta v_A}{\Delta v_P} &= u\Bigg[\tfrac{1}{\sqrt{2}-1}\left(1-\sqrt{2}u\right)+\tfrac{1}{2}\tfrac{\sqrt{2}}{\left(\sqrt{2}-1\right)^2}\left(u^2-u^3\right) \\ &\quad +\tfrac{1}{8}\tfrac{\sqrt{2}\left(3-\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^3}\left(u^4-u^5\right)+\tfrac{1}{16}\tfrac{\sqrt{2}\left(7-4\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^4}\left(u^5-u^6\right) \\ &\quad +\tfrac{1}{128}\tfrac{\sqrt{2}\left(85-57\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^5}\left(u^7-u^8\right)+\tfrac{1}{256}\tfrac{\sqrt{2}\left(295-206\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^6}\left(u^9-u^{10}\right)+\cdots\Bigg]\end{aligned} \tag{34}$$

where we have set $u = \sqrt{\varepsilon} = \sqrt{a_1/a_2}$. We carry out the expansion to an impractical number of terms in order to show the interesting pattern of the expansion coefficients.

7 Effects of Errors in the Velocity Changes

7.1 Impulse Error at Pericenter

7.1.1 Perturbed Transfer Orbit Elements

Suppose, as is always the case in the real world, an error occurs and the change in velocity magnitude imparted by the first thruster firing is in error by some small amount, say $\Delta v_P = \Delta v_P^0 + \delta v_P$, where Δv_P^0 is the desired change in velocity given by (22) and δv_P is the error. What are the effects on the transfer orbit semimajor axis and eccentricity, and on the outer orbit?

Using (22), we find the variation in Δv_P is

$$\delta\Delta v_P = \delta v_P = \frac{\partial \Delta v_P}{\partial a_2}\delta a_2 = \sqrt{\frac{\mu}{2}\frac{a_1}{a_2}\frac{1}{\left(a_1+a_2\right)^3}}\delta a_2 \tag{35}$$

Thus, we can write the resulting error in the radius of the resulting circular orbit (assuming, ideally, that a compensating adjustment of the thrust at A circularizes it),

$$\delta a_2 = \left[\frac{\mu}{2}\frac{a_1}{a_2}\frac{1}{\left(a_1+a_2\right)^3}\right]^{-1/2}\delta v_P \tag{36}$$

or

$$\frac{\delta a_2}{a_2} = \sqrt{\frac{2(1+\varepsilon)^3}{\varepsilon}}\frac{\delta v_P}{v_c(A)} = \sqrt{\frac{2}{\varepsilon}}\left(1+\frac{3}{2}\varepsilon+\frac{3}{8}\varepsilon^2-\frac{1}{16}\varepsilon^3+\cdots\right)\frac{\delta v_P}{v_c(A)} \tag{37}$$

where $v_c(A) = \sqrt{\mu/a_2}$ is the unperturbed outer orbit circular velocity and $\varepsilon = a_1/a_2$.

From eq. (2) for e, we have the error in the transfer orbit eccentricity,

$$\delta e = \frac{2a_1}{(a_1+a_2)^2}\delta a_2 = \frac{2\varepsilon}{(1+\varepsilon)^2}\frac{\delta a_2}{a_2} = 2\big(\varepsilon - 2\varepsilon^2 + 3\varepsilon^3 - 4\varepsilon^4 + \cdots\big)\frac{\delta a_2}{a_2} \tag{38}$$

while the corresponding error in the transfer orbit semimajor axis is

$$\delta a = \frac{1}{2}\delta a_2 \tag{39}$$

7.1.2 Circularize the Outer Orbit at the Perturbed Radius

What is the impulse required to circularize the outer orbit? That would be the difference in velocities between the circular orbit of perturbed radius $a_2 + \delta a_2$ and the apocenter of the perturbed transfer ellipse of semimajor axis $a + \delta a$. Thus, from (24) and (26), we may write

$$\Delta v_A = \sqrt{\frac{\mu}{a_2+\delta a_2}} - \sqrt{\frac{2\mu}{a_1+a_2+\delta a_2}\frac{a_1}{a_2+\delta a_2}} \tag{40}$$

which, to first order, becomes

$$\Delta v_A = \Delta v_A^0 - \delta\Delta v_A = \sqrt{\frac{\mu}{a_2}}\left[1 - \sqrt{\frac{2a_1}{a_1+a_2}} - \left(1 - \sqrt{\frac{2a_1}{a_1+a_2}\frac{a_1+2a_2}{a_1+a_2}}\right)\frac{\delta a_2}{2a_2} + \cdots\right] \tag{41}$$

or

$$\Delta v_A = \sqrt{\frac{\mu}{a_2}}\left[1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}} - \left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}\frac{2+\varepsilon}{1+\varepsilon}}\right)\frac{\delta a_2}{2a_2} + \cdots\right] \tag{42}$$

where Δv_A^0 is the unperturbed circularizing impulse, eq. (23). Hence, the adjustment impulse — the small change in velocity that we must subtract from the unperturbed velocity change given by (23) — is, to first order,

$$\delta\Delta v_A = \frac{1}{2}\sqrt{\frac{\mu}{a_2}}\left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}\frac{2+\varepsilon}{1+\varepsilon}}\right)\frac{\delta a_2}{a_2} + \cdots \tag{43}$$

Substituting eq. (36) for δa_2, we can express the corrections in terms of the original velocity error at pericenter. Then eq. (42) becomes

$$\frac{\Delta v_A}{v_c(A)} = \left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}}\right) - \left[\sqrt{\frac{1+\varepsilon}{2\varepsilon}}(1+\varepsilon) - (2+\varepsilon)\right]\frac{\delta v_P}{v_c(A)} + \cdots \tag{44}$$

Another way to achieve the same results is to consider instead the variation of Δv_A when a_2 varies. From (24), (26), and (28),

$$\begin{aligned} -\delta\Delta v_A &= -\tfrac{\partial \Delta v_A}{\partial a_2}\delta a_2 \\ &= -\delta a_2 \tfrac{\partial}{\partial a_2}\left[\sqrt{\tfrac{\mu}{a_2}}\left(1 - \sqrt{\tfrac{2a_1}{a_1+a_2}}\right)\right] \\ &= \sqrt{\tfrac{\mu}{a_2}}\left(1 - \sqrt{\tfrac{2a_1}{a_1+a_2}\tfrac{a_1+2a_2}{a_1+a_2}}\right)\tfrac{\delta a_2}{2a_2} \end{aligned} \tag{45}$$

which we see is identical to (43). Yet a third way is to consider eq. (26) and plug in the variations directly. We have

$$v_A = \sqrt{\frac{\mu}{a}\frac{1-e}{1+e}} = \sqrt{\frac{\mu}{a+\delta a}\frac{1-e-\delta e}{1+e+\delta e}} \tag{46}$$

for the perturbed transfer orbit aphelion velocity. Use (1) and (2) for a and e, then (38) and (39) for δe and δa. One finds

$$v_A = \sqrt{\frac{2\mu}{a_1+a_2+\delta a_2}\frac{a_1(a_1+a_2)-a_1\delta a_2}{a_2(a_1+a_2)+a_1\delta a_2}} = \sqrt{\frac{2\mu}{a_1+a_2}\frac{a_1}{a_2}}\left(1-\frac{a_1+2a_2}{a_1+a_2}\frac{\delta a_2}{2a_2}+\cdots\right) \tag{47}$$

The circular velocity at the perturbed radius is

$$v_c(a_2+\delta a_2) = \sqrt{\frac{\mu}{a_2+\delta a_2}} = \sqrt{\frac{\mu}{a_2}}\left(1-\frac{\delta a_2}{2a_2}+\cdots\right) \tag{48}$$

The difference is then the circularization impulse:

$$\Delta v_A = v_c(a_2+\delta a_2) - v_A \tag{49}$$

Upon substituting (47) and (48), one finds again eq. (41).

7.1.3 Eliminating the First-Order Term

We see from (43) and (44) that for some value of the ratio a_1/a_2 the first-order correction will be zero. Solving the expression in brackets in (44) is equivalent to solving

$$2\,(1+\varepsilon)^3 = \varepsilon\,(4+2\varepsilon)^2 \tag{50}$$

Eq. (50) has three real solutions with two of them being negative and hence unphysical. Thus, the adjustment $\delta\Delta v_A$ is a positive quantity when

$$\frac{a_1}{a_2} < \frac{2}{3}\sqrt{10}\cos\left(\frac{1}{3}\arctan\left(3\sqrt{111}\right)\right) - \frac{5}{3} = 0.170086\ldots \tag{51}$$

Therefore, oddly enough, if the ratio of unperturbed circular orbits is the value given by equality in eq. (51), then the adjustment impulse is second order in the perturbation δa_2. Numerically, the expansion (41) becomes

$$\Delta v_A = v_c(A)\left[0.46081 - 0.09008\left(\frac{\delta a_2}{a_2}\right)^2 + 0.12142\left(\frac{\delta a_2}{a_2}\right)^3 - \cdots\right] \tag{52}$$

and eq. (44) becomes

$$\Delta v_A = v_c(A)\left[0.46081 - 1.69691\left(\frac{\delta v_P}{v_c(A)}\right)^2 + 9.92701\left(\frac{\delta v_P}{v_c(A)}\right)^3 - \cdots\right] \tag{53}$$

where $v_c(A) = \sqrt{\mu/a_2}$ is the unperturbed outer orbit circular velocity.

7.1.4 An Eccentric Perturbed Outer Orbit

The foregoing consider the consequences that result from an error δv at the pericenter point P of the transfer orbit, assuming the Δv_A impulse is adjusted to compensate so that we end at A with a circular orbit now of radius $a_2 + \delta a_2$. If the velocity error at P is not detected or for some other reason the impulse at A is not adjusted to compensate, then application of the unperturbed Δv_A from eq. (23) will result in an eccentric orbit. Now, the velocity at the apocenter of the perturbed transfer orbit is given to first order by (47). Application of the planned thrust Δv_A^0 at the apocenter, eq. (23), results in the new velocity magnitude

$$v = v_A + \Delta v_A^0 \tag{54}$$

We find

$$v = \sqrt{\frac{\mu}{a_2}}\left(1 - \sqrt{\frac{2a_1}{a_1+a_2}}\frac{a_1+2a_2}{a_1+a_2}\frac{\delta a_2}{2a_2} + \cdots\right) = v_c(A)\left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}}\frac{2+\varepsilon}{1+\varepsilon}\frac{\delta a_2}{2a_2} + \cdots\right) \tag{55}$$

This velocity is now the pericenter or apocenter of the final orbit, which is an ellipse.

We would like to know the orbital elements of this ellipse. The apocenter or pericenter distance is, respectively, $r = a(1 \pm e)$. Thus, we can write

$$a(1 \pm e) = a_2 + \delta a_2 \tag{56}$$

From the vis viva integral (7), we have

$$v^2 = \mu\left(\frac{2}{a(1\pm e)} - \frac{1}{a}\right) = \frac{\mu}{a}\frac{1 \mp e}{1 \pm e} \tag{57}$$

Now we may solve (56) and (57) to get

$$\pm e = 1 - \frac{a_2 + \delta a_2}{\mu} v^2 \tag{58}$$

and

$$a = \frac{a_2 + \delta a_2}{2 - \frac{v^2}{\mu}(a_2 + \delta a_2)} \tag{59}$$

Using (55) for v, we find the first-order results

$$\pm e = \left(\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\frac{2+\varepsilon}{1+\varepsilon} - 1\right)\frac{\delta a_2}{a_2} + \cdots = \left(2\sqrt{2\varepsilon} - 1 + \cdots\right)\frac{\delta a_2}{a_2} + \cdots \tag{60}$$

and

$$\frac{a}{a_2} = 1 + \left(2 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}}\frac{2+\varepsilon}{1+\varepsilon}\right)\frac{\delta a_2}{a_2} + \cdots = 1 + 2\left(1 - \sqrt{2\varepsilon} + \cdots\right)\frac{\delta a_2}{a_2} + \cdots \tag{61}$$

If the right hand side of (60) is positive/negative, the thruster firing point is the final ellipse apocenter/pericenter.

7.2 Impulse Error at Apocenter

Suppose the impulse at P occurs as planned, but the thrust at the transfer orbit apocenter has an error. An impulse error at A, δv_A, will cause the outer orbit to be eccentric rather than circular. The velocity magnitude after the erroneous thruster firing is then

$$v = \Delta v_A + \delta v_A \tag{62}$$

where the unperturbed circularizing impulse Δv_A is given by (23). Similar to eq. (56), we have

$$a(1 \pm e) = a_2 \tag{63}$$

while eq. (57) still holds. Solving (57) and (63), we have

$$\pm e = 1 - \frac{a_2}{\mu} v^2 = 1 - \left(\frac{v}{v_c(A)} \right)^2 \tag{64}$$

and

$$\frac{a}{a_2} = \frac{1}{2 - \frac{a_2}{\mu} v^2} = \frac{1}{2 - \left(\frac{v}{v_c(A)} \right)^2} \tag{65}$$

Substituting (62) for v and (23) for Δv_A, we find

$$\pm e = 1 - \left(\frac{\Delta v_A + \delta v_A}{v_c(A)} \right)^2 = 1 - \left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}} \right)^2 - 2 \left(1 - \sqrt{\frac{2\varepsilon}{1+\varepsilon}} \right) \frac{\delta v_A}{v_c(A)} - \left(\frac{\delta v_A}{v_c(A)} \right)^2 \tag{66}$$

(exact) and

$$\begin{aligned} \frac{a}{a_2} &= \frac{1}{2 - \left(\frac{\Delta v_A}{v_c(A)} \right)^2} + \frac{\Delta v_A v_c^2(A)}{\left(2v_c^2(A) - \Delta v_A^2 \right)^2} \delta v_A + \cdots \\ &= \frac{1+\varepsilon}{1 + 2\sqrt{2\varepsilon(1+\varepsilon)} - \varepsilon} + 2(1+\varepsilon) \frac{1 - 2\sqrt{2\varepsilon(1+\varepsilon)} + \varepsilon}{\left(1 + 2\sqrt{2\varepsilon(1+\varepsilon)} - \varepsilon \right)^2} \frac{\delta v_A}{v_c(A)} + \cdots \end{aligned} \tag{67}$$

where $v_c(A) = \sqrt{\mu/a_2}$ and (67) is to first order in δv_A. If the ratio $\varepsilon = a_1/a_2$ is small, then

$$\pm e = 2\left(\sqrt{2\varepsilon} - \varepsilon + \cdots \right) - 2\left(1 - \sqrt{2\varepsilon} + \cdots \right) \frac{\delta v_A}{v_c(A)} - \left(\frac{\delta v_A}{v_c(A)} \right)^2 \tag{68}$$

and

$$\frac{a}{a_2} = \left(1 - 2\sqrt{2\varepsilon} + 10\varepsilon + \cdots \right) + 2\left(1 - 5\sqrt{2\varepsilon} + 36\varepsilon + \cdots \right) \frac{\delta v_A}{v_c(A)} + \cdots \tag{69}$$