Partial Derivatives of Observables with Respect to Two-Body Orbital Elements

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Abstract

We calculate the partial derivatives of the observable quantities $\{\alpha, \delta, r, \dot{r}\}$, where α is the right ascension and δ is the declination, with respect to the two-body orbital elements $\{a, e, \iota, \Omega, \omega, E\}$. The observables are those that are available, at least in principle, when the goal is the determination of orbital elements for orbiting bodies. We avoid the complexity of expressing the observables directly in terms of orbital elements by instead expressing them in Cartesian coordinates and then invoking the chain rule for derivatives.

Subject headings: celestial mechanics—two-body problem

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1 Components of the Two-Body State Vector

1.1 The Position Vector

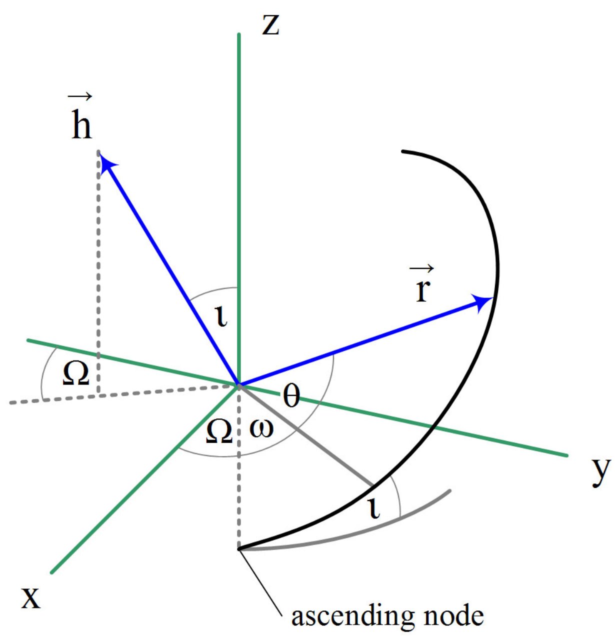

Figure 1: The orbital elements and angular momentum vector.

Figure 1 shows an orbit with the standard set of orbital elements $\{a, e, \iota, \omega, \Omega, \theta\}$, where θ is the true anomaly and $\overrightarrow{h}$ is the angular momentum (per mass). To reduce expression complexity, it is useful to use the eccentric anomaly E as the fast angular variable instead of the true anomaly. An unperturbed two-body elliptical orbit radius vector can be written

$$\overrightarrow{r}(E) = a\mathcal{Q}(\Omega, \iota, \omega) \cdot \overrightarrow{q}(E) \tag{1}$$

where a is the semimajor axis, e the orbital eccentricity, ι the orbital inclination, Ω the longitude of the ascending node, and ω the argument of pericenter. The position/shape vector is

$$\overrightarrow{q}(E) = \left\langle \cos E - e, \sqrt{1-e^2}\sin E, 0 \right\rangle \tag{2}$$

and the orbit orientation matrix is

$$\mathcal{Q}(\Omega, \iota, \omega) = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos\iota & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos\iota & \sin\Omega\sin\iota \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos\iota & -\cos\Omega\sin\iota \\ \sin\omega\sin\iota & \cos\omega\sin\iota & \cos\iota \end{bmatrix} \tag{3}$$

where

$$\mathcal{Q}\left(\Omega,\iota,\omega\right)=\mathcal{R}_z(\Omega)\cdot\mathcal{R}_x(\iota)\cdot\mathcal{R}_z(\omega) \tag{4}$$

is the composition of the rotation matrices

$$\mathcal{R}_x\left(\psi\right)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sin\psi & \cos\psi\end{bmatrix} \tag{5}$$

$$\mathcal{R}_z\left(\psi\right)=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{6}$$

The matrices (5) and (6) individually rotate a vector counterclockwise around the respective x or z coordinate axis of an inertial frame. (Alternatively, one may interpret these as rotating the coordinate frame clockwise about the respective axes.)

The formulation (1) cleanly separates the orbit scale a, the shape of the orbit plus the position of the particle at a given time $\overrightarrow{q}\,(t)$ (with $E(t)$ obtained via the Kepler equation), and the orientation of the orbit in space $\mathcal{Q}\left(\Omega,\iota,\omega\right)$.

1.2 Scalar Distance

The matrix $\mathcal{Q}\left(\Omega,\iota,\omega\right)$ is orthogonal, since it is composed of orthogonal matrices. Hence, $\mathcal{Q}\left(\Omega,\iota,\omega\right)^\top\cdot\mathcal{Q}\left(\Omega,\iota,\omega\right)=\mathcal{I}$, where $\mathcal{I}$ is the unit matrix. The magnitude of the radius vector can therefore formally be derived from (1) as

$$\begin{aligned} r^2 &\equiv \overrightarrow{r}^\top\cdot\overrightarrow{r}\\ &= a^2\left(\overrightarrow{q}^\top\cdot\mathcal{Q}^\top\right)\cdot\left(\mathcal{Q}\cdot\overrightarrow{q}\right)\\ &= a^2\overrightarrow{q}^\top\cdot\left(\mathcal{Q}^\top\cdot\mathcal{Q}\right)\cdot\overrightarrow{q}\\ &= a^2\left(\overrightarrow{q}^\top\cdot\mathcal{I}\cdot\overrightarrow{q}\right)\\ &= a^2\left(\overrightarrow{q}^\top\cdot\overrightarrow{q}\right)\\ &= a^2q^2 \end{aligned} \tag{7}$$

From the definition (2), $q^2=(1-e\cos E)^2$. Thus, the scalar distance is

$$r\left(E\right)=\sqrt{\overrightarrow{r}\,(E)^\top\cdot\overrightarrow{r}\,(E)}=a\sqrt{\overrightarrow{q}\,(E)^\top\cdot\overrightarrow{q}\,(E)}=a(1-e\cos E) \tag{8}$$

1.3 Radial Velocity

The radial velocity is the time derivative of the magnitude of the position vector. If we write the velocity vector in radial and perpendicular parts,

$$\overrightarrow{v}=\dot{r}\widehat{r}+r\dot{\theta}\widehat{\theta} \tag{9}$$

One immediately sees that the projection onto the line of sight of an observer at the coordinate origin is

$$\overrightarrow{v}\cdot\widehat{r}=\dot{r} \tag{10}$$

We shall use (8) to calculate $\dot{r}$. We see that we shall first need $\dot{E}$. Thus, take the time derivative of the Kepler equation

$$E - e\sin E = M = n(t - \tau) \tag{11}$$

where n is the mean motion and τ is the time of pericenter passage, to get

$$\dot{E}(1 - e\cos E) = n \tag{12}$$

which by (8) is just

$$\dot{E} = \frac{na}{r} \tag{13}$$

Differentiating (8), we then find our result,

$$\dot{r} = \frac{na^2}{r} e\sin E = \frac{na}{1 - e\cos E} e\sin E \tag{14}$$

1.4 Velocity Vector

The two-body velocity is, by differentiating (1),

$$\overrightarrow{v} = a\mathcal{Q}(\Omega, \iota, \omega) \cdot \frac{\partial \overrightarrow{q}(E)}{\partial E} \dot{E} \tag{15}$$

From (2), (8), and (13) follows the result

$$\begin{aligned} \overrightarrow{v} &= \tfrac{na}{1-e\cos E} \mathcal{Q}(\Omega, \iota, \omega) \cdot \tfrac{\partial \overrightarrow{q}(E)}{\partial E} \\ &= \tfrac{na}{1-e\cos E} \mathcal{Q}(\Omega, \iota, \omega) \cdot \begin{bmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{bmatrix} \end{aligned} \tag{16}$$

2 Partial Derivatives of the State Vector

2.1 Position

From (1) it is straightforward to calculate the partial derivatives of the position vector with respect to the orbital parameters $\{a, e, \iota, \Omega, \omega, E\}$. We find

$$\begin{aligned} \tfrac{\partial \overrightarrow{r}}{\partial a} &= \mathcal{Q}(\Omega, \iota, \omega) \cdot \overrightarrow{q}(E) \\ &= \tfrac{\overrightarrow{r}}{a} \\ \tfrac{\partial \overrightarrow{r}}{\partial e} &= a\mathcal{Q}(\Omega, \iota, \omega) \cdot \tfrac{\partial \overrightarrow{q}(E)}{\partial e} \\ &= -a\mathcal{Q}(\Omega, \iota, \omega) \cdot \left\langle 1, \tfrac{e\sin E}{\sqrt{1-e^2}}, 0 \right\rangle \\ \tfrac{\partial \overrightarrow{r}}{\partial \{\Omega,\iota,\omega\}} &= a\tfrac{\partial \mathcal{Q}(\Omega,\iota,\omega)}{\partial \{\Omega,\iota,\omega\}} \cdot \overrightarrow{q}(E) \\ \tfrac{\partial \overrightarrow{r}}{\partial E} &= a\mathcal{Q}(\Omega, \iota, \omega) \cdot \tfrac{\partial \overrightarrow{q}(E)}{\partial E} \\ &= a\mathcal{Q}(\Omega, \iota, \omega) \cdot \left\langle -\sin E, \sqrt{1-e^2}\cos E, 0 \right\rangle \end{aligned} \tag{17}$$

2.2 Velocity

Recall that $\mu = n^2 a^3$ (Kepler's third law), where $\mu = G(m_1 + m_2)$. Thus, for the quantity na that appears in the velocity terms (cf. eq. (16)), we have

$$na = \sqrt{\frac{\mu}{a}} \tag{18}$$

One must be careful to not naively take $\partial(na)/\partial a = n$. Right away, something must be wrong with that, since na is a velocity (the circular velocity of an orbit with radius a), and physical intuition indicates that an increase in semimajor axis should cause na to *decrease*. Instead we must take the derivative of the right-hand side of (18). Hence,

$$\frac{\partial(na)}{\partial a} = \frac{\partial}{\partial a}\sqrt{\frac{\mu}{a}} = -\frac{1}{2}\sqrt{\frac{\mu}{a^3}} = -\frac{1}{2}n \tag{19}$$

The partials of the velocity vector (16) then become

$$\begin{aligned}
\frac{\partial \vec{v}}{\partial a} &= -\frac{1}{2}\frac{n}{1-e\cos E}\mathcal{Q}(\Omega,\iota,\omega)\cdot\frac{\partial \vec{q}(E)}{\partial E} \\
&= -\frac{1}{2}\frac{\vec{v}}{a} \\
\frac{\partial \vec{v}}{\partial e} &= \frac{na}{1-e\cos E}\mathcal{Q}(\Omega,\iota,\omega)\cdot\left(\frac{\cos E}{1-e\cos E}\frac{\partial \vec{q}(E)}{\partial E} + \frac{\partial^2 \vec{q}(E)}{\partial E\partial e}\right) \\
&= \frac{na}{1-e\cos E}\mathcal{Q}(\Omega,\iota,\omega)\cdot\left(\frac{\cos E}{1-e\cos E}\begin{bmatrix}-\sin E \\ \sqrt{1-e^2}\cos E \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ \frac{-e\cos E}{\sqrt{1-e^2}} \\ 0\end{bmatrix}\right) \\
\frac{\partial \vec{v}}{\partial\{\Omega,\iota,\omega\}} &= \frac{na}{1-e\cos E}\frac{\partial\mathcal{Q}(\Omega,\iota,\omega)}{\partial\{\Omega,\iota,\omega\}}\cdot\frac{\partial \vec{q}(E)}{\partial E} \\
&= \frac{na}{1-e\cos E}\frac{\partial\mathcal{Q}(\Omega,\iota,\omega)}{\partial\{\Omega,\iota,\omega\}}\cdot\left\langle -\sin E, \sqrt{1-e^2}\cos E, 0\right\rangle \\
\frac{\partial \vec{v}}{\partial E} &= \frac{-na}{1-e\cos E}\mathcal{Q}(\Omega,\iota,\omega)\cdot\left(\frac{e\sin E}{1-e\cos E}\frac{\partial \vec{q}(E)}{\partial E} + \frac{\partial^2 \vec{q}(E)}{\partial E^2}\right) \\
&= \frac{-na}{1-e\cos E}\mathcal{Q}(\Omega,\iota,\omega)\cdot\left(\frac{e\sin E}{1-e\cos E}\begin{bmatrix}-\sin E \\ \sqrt{1-e^2}\cos E \\ 0\end{bmatrix} + \begin{bmatrix}\cos E \\ \sqrt{1-e^2}\sin E \\ 0\end{bmatrix}\right)
\end{aligned} \tag{20}$$

For completeness, in the Appendix we explicitly expand eqs. (17) and (20) to component form. A file containing C language subroutines, optimized with regard to cpu time, that numerically calculate the derivatives in eqs. (17) and (20) is available at
http://www.alpheratz.net/murison/dynamics/twobody/twobody_partials.c.

3 Observables and the State Vector

When observing orbiting bodies in the Solar System, we potentially have up to four observable quantities. The angular position on the plane of the sky is specified by two observables, say right ascension and declination, and is provided by astrometry. The radial velocity can be had from spectroscopy or doppler radar. Finally, the distance can be found by radar ranging. In Section 4, we will calculate the partial derivatives of these observables with respect to the orbital elements. First, though, it will be convenient to determine the partial derivatives of the observables with respect to the Cartesian state vector

$$\overrightarrow{S} = \langle x, y, z, \dot{x}, \dot{y}, \dot{z} \rangle \tag{21}$$

as an intermediate step, and then invoke the chain rule for derivatives. This approach is more useful because it would be difficult to express the observables directly as functions of the orbital elements. Thus, we need to relate explicitly the observables to the components of $\overrightarrow{S}$.

3.1 Astrometric Angles

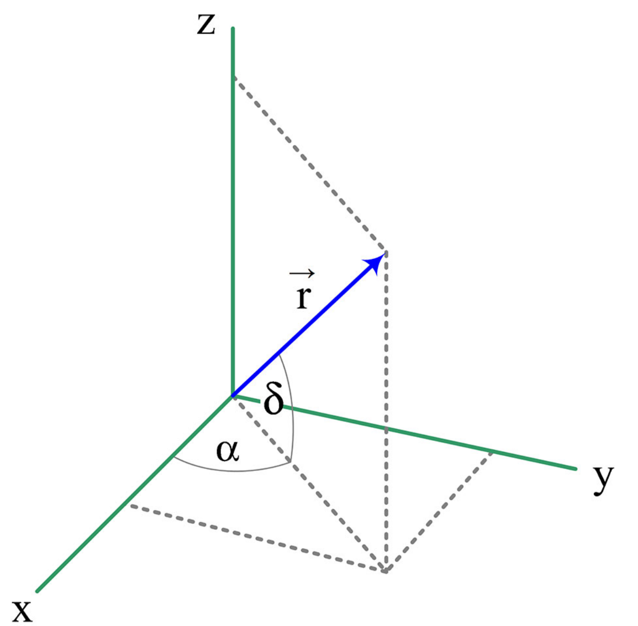

Figure 2: Right ascension α and declination δ.

From Figure 2 we see that

$$\begin{aligned} x &= r\cos\delta\cos\alpha \\ y &= r\cos\delta\sin\alpha \\ z &= r\sin\delta \end{aligned} \tag{22}$$

where α is the right ascension and δ is the declination. We can write (22) compactly as the unit vector

$$\frac{\overrightarrow{r}}{r} = \overrightarrow{F}(\alpha, \delta) \tag{23}$$

The inverse is then

$$\langle \alpha, \delta \rangle = \overrightarrow{G}(\overrightarrow{r}) \tag{24}$$

From (22) (or by inspection of Figure 2), eq. (24) becomes

$$\begin{aligned} \sin\delta &= \tfrac{z}{r} \\ \sin\alpha &= \tfrac{y/r}{\sqrt{1-(z/r)^2}} = \tfrac{y}{\sqrt{x^2+y^2}} \\ \cos\alpha &= \tfrac{x}{\sqrt{1-(z/r)^2}} = \tfrac{x}{\sqrt{x^2+y^2}} \end{aligned} \tag{25}$$

We therefore can easily convert back and forth between the position vector and the angular observables, and from (25) we can calculate the Cartesian derivatives of $\{\alpha, \delta\}$.

3.2 Scalar Distance

The scalar distance is just

$$r = \sqrt{x^2 + y^2 + z^2} \tag{26}$$

3.3 Radial Velocity

The radial velocity can be written

$$\dot{r} = \overrightarrow{v} \cdot \widehat{r} = \frac{\overrightarrow{v} \cdot \overrightarrow{r}}{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \tag{27}$$

4 Partials of the Observables

Let $p \in \{a, e, \iota, \Omega, \omega, E\}$ be an orbital parameter, and let $\varphi \in \{\alpha, \delta, r, \dot{r}\}$ be an observable. Then, in general, the partial derivatives of the observables with respect to the orbital elements can be written

$$\frac{\partial \varphi}{\partial p} = \sum_{k=1}^{6} \frac{\partial \varphi}{\partial \left[\overrightarrow{S}\right]_k} \frac{\partial \left[\overrightarrow{S}\right]_k}{\partial p} \tag{28}$$

where $\left[\overrightarrow{S}\right]_k$ is the k^{th} component of the Cartesian state vector, eq. (21). Equation (28) is the fundamental relation. The derivatives $\frac{\partial \left[\overrightarrow{S}\right]_k}{\partial p}$ are given by eqs. (17) and (20). We shall now calculate the partials of the observables with respect to the Cartesian state vector components, $\frac{\partial \varphi}{\partial \left[\overrightarrow{S}\right]_k}$. Once that is done, then eq. (28) can be evaluated for the partial derivatives of each observable with respect to each of the orbital elements.

4.1 Astrometric Angles

From eqs. (25), we find that

$$\begin{aligned} \frac{\partial\alpha}{\partial x} &= -\frac{y}{x^2+y^2} \\ \frac{\partial\alpha}{\partial y} &= \frac{x}{x^2+y^2} \\ \frac{\partial\alpha}{\partial z} &= 0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} \frac{\partial\delta}{\partial x} &= -\frac{xz}{r^2}\frac{1}{\sqrt{x^2+y^2}} \\ \frac{\partial\delta}{\partial y} &= -\frac{yz}{r^2}\frac{1}{\sqrt{x^2+y^2}} \\ \frac{\partial\delta}{\partial z} &= \frac{\sqrt{x^2+y^2}}{r^2} \end{aligned} \tag{30}$$

4.2 Scalar Distance

From (26) we have

$$\frac{\partial r}{\partial\{x,y,z\}} = \frac{\{x,y,z\}}{r} \tag{31}$$

4.3 Radial Velocity

From (27) we calculate

$$\frac{\partial\dot{r}}{\partial\{x,y,z\}} = \frac{\{\dot{x},\dot{y},\dot{z}\}}{r} - \frac{\dot{r}}{r^2}\{x,y,z\} \tag{32}$$

and

$$\frac{\partial\dot{r}}{\partial\{\dot{x},\dot{y},\dot{z}\}} = \frac{\{x,y,z\}}{r} \tag{33}$$

5 Appendix: Explicit State Vector Derivatives

Recall that the state vector in Cartesian components is

$$\overrightarrow{S} = \langle x, y, z, \dot{x}, \dot{y}, \dot{z}\rangle \tag{34}$$

From eqs. (17) and (20) we explicitly calculate the components of the partial derivatives of $\overrightarrow{S}$ with respect to the orbital elements $\{a, e, \iota, \Omega, \omega, E\}$. Optimized C code subroutines for calculating these derivatives numerically is available at
http://www.alpheratz.net/murison/dynamics/twobody/twobody_partials.c.

$$\frac{\partial \overrightarrow{S}}{\partial a} = \begin{bmatrix} -\sin E(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\sqrt{1-e^2} + (\cos E - e)(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos\iota) \\ -\sin E(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos\iota)\sqrt{1-e^2} + (\cos E - e)(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota) \\ \sin E\cos\omega\sqrt{1-e^2} + (\cos E - e)\sin\omega\sin\iota \\ \frac{1}{2}\frac{n}{1-e\cos E}\left[\cos E(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\sqrt{1-e^2} + \sin E(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos\iota)\right] \\ \frac{1}{2}\frac{n}{1-e\cos E}\left[-\cos E(-\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos\iota)\sqrt{1-e^2} + \sin E(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota)\right] \\ \frac{1}{2}\frac{n}{1-e\cos E}\left[\left(\sin\omega\sin E - \cos\omega\cos E\sqrt{1-e^2}\right)\sin\iota\right] \end{bmatrix} \tag{35}$$

$$\frac{\partial \overrightarrow{S}}{\partial e} = \begin{bmatrix} a\left(-\cos\Omega\cos\omega + \sin\Omega\sin\omega\cos\iota + \frac{e}{\sqrt{1-e^2}}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\sin E\right) \\ a\left(-\sin\Omega\cos\omega - \cos\Omega\sin\omega\cos\iota + \frac{e}{\sqrt{1-e^2}}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos\iota)\sin E\right) \\ -a\left(\sin\omega + \frac{e}{\sqrt{1-e^2}}\cos\omega\sin E\right)\sin\iota \\ \frac{na\cos E}{(1-e\cos E)^2}\left[(-\cos\Omega\cos\omega + \sin\Omega\sin\omega\cos\iota)\sin E - \frac{\cos E - e}{\sqrt{1-e^2}}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\right] \\ \frac{na\cos E}{(1-e\cos E)^2}\left[-(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota)\sin E - \frac{\cos E - e}{\sqrt{1-e^2}}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos\iota)\right] \\ \frac{na\cos E}{(1-e\cos E)^2}\left[-\sin\omega\sin E + \frac{\cos E - e}{\sqrt{1-e^2}}\cos\omega\right] \end{bmatrix} \tag{36}$$

$$\frac{\partial \overrightarrow{S}}{\partial \iota} = \begin{bmatrix} a\left[\sqrt{1-e^2}\cos\omega\sin E + (\cos E - e)\sin\omega\right]\sin\Omega\sin\iota \\ -a\left[\sqrt{1-e^2}\cos\omega\sin E + (\cos E - e)\sin\omega\right]\cos\Omega\sin\iota \\ a\left[\sqrt{1-e^2}\cos\omega\sin E + (\cos E - e)\sin\omega\right]\cos\iota \\ \frac{-na}{1-e\cos E}\left(\sin\omega\sin E - \sqrt{1-e^2}\cos\omega\cos E\right)\sin\Omega\sin\iota \\ \frac{na}{1-e\cos E}\left(\sin\omega\sin E - \sqrt{1-e^2}\cos\omega\cos E\right)\cos\Omega\sin\iota \\ \frac{-na}{1-e\cos E}\left(\sin\omega\sin E - \sqrt{1-e^2}\cos\omega\cos E\right)\cos\iota \end{bmatrix} \tag{37}$$

$$\frac{\partial \overrightarrow{S}}{\partial \Omega} = \begin{bmatrix} a\left[(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\sin E - (\cos E - e)(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota)\right] \\ a\left[-(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\sin E + (\cos E - e)(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos\iota)\right] \\ 0 \\ \frac{na}{1-e\cos E}\left[(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos\iota)\sin E + \sqrt{1-e^2}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos\iota)\cos E\right] \\ \frac{na}{1-e\cos E}\left[(-\cos\Omega\cos\omega + \sin\Omega\sin\omega\cos\iota)\sin E - \sqrt{1-e^2}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos\iota)\cos E\right] \\ 0 \end{bmatrix} \tag{38}$$

$$
\frac{\partial\overrightarrow{S}}{\partial\omega}=\begin{bmatrix}
a\Big[(-\cos\Omega\cos\omega+\sin\Omega\sin\omega\cos\iota)\sqrt{1-e^2}\sin E-(\cos E-e)(\cos\Omega\sin\omega+\sin\Omega\cos\omega\cos\iota)\Big]\\
-a\Big[(\sin\Omega\cos\omega+\cos\Omega\sin\omega\cos\iota)\sqrt{1-e^2}\sin E+(\cos E-e)(\sin\Omega\sin\omega-\cos\Omega\cos\omega\cos\iota)\Big]\\
a\Big[-\sqrt{1-e^2}\sin\omega\sin E+(\cos E-e)\cos\omega\Big]\sin\iota\\
\frac{na}{1-e\cos E}\Big[(\cos\Omega\sin\omega+\sin\Omega\cos\omega\cos\iota)\sin E-\sqrt{1-e^2}(\cos\Omega\cos\omega-\sin\Omega\sin\omega\cos\iota)\cos E\Big]\\
\frac{na}{1-e\cos E}\Big[(\sin\Omega\sin\omega-\cos\Omega\cos\omega\cos\iota)\sin E-\sqrt{1-e^2}(\sin\Omega\cos\omega+\cos\Omega\sin\omega\cos\iota)\cos E\Big]\\
\frac{-na}{1-e\cos E}\Big(\cos\omega\sin E+\sqrt{1-e^2}\sin\omega\cos E\Big)\sin\iota
\end{bmatrix} \tag{39}
$$

$$
\frac{\partial\overrightarrow{S}}{\partial E}=\begin{bmatrix}
-a\Big[(\cos\Omega\sin\omega+\sin\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\cos E+(\cos\Omega\cos\omega-\sin\Omega\sin\omega\cos\iota)\sin E\Big]\\
-a\Big[(\sin\Omega\sin\omega-\cos\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\cos E+(\sin\Omega\cos\omega+\cos\Omega\sin\omega\cos\iota)\sin E\Big]\\
a\Big(\sqrt{1-e^2}\cos\omega\cos E-\sin\omega\sin E\Big)\sin\iota\\
\frac{na}{(1-e\cos E)^2}\Big[(\cos\Omega\sin\omega+\sin\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\sin E+(\cos E-e)(-\cos\Omega\cos\omega+\sin\Omega\sin\omega\cos\iota)\Big]\\
\frac{na}{(1-e\cos E)^2}\Big[(\sin\Omega\sin\omega-\cos\Omega\cos\omega\cos\iota)\sqrt{1-e^2}\sin E-(\cos E-e)(\sin\Omega\cos\omega+\cos\Omega\sin\omega\cos\iota)\Big]\\
\frac{-na}{(1-e\cos E)^2}\Big[\sqrt{1-e^2}\cos\omega\sin E+(\cos E-e)\sin\omega\Big]\sin\iota
\end{bmatrix} \tag{40}
$$